Analytic Solution for Low Thrust Out of Ecliptic Mission Design

G.A. Flandro*
Jet Propulsion Laboratory, Pasadena, Calif.
and University of Utah, Salt Lake City, Utah

This paper describes an analytical trajectory solution for low thrust out of ecliptic mission design based on the two-variable asymptotic expansion technique. A practical steering program utilizing normal thrusting symmetrical about the nodes is assumed. A two-term expansion yields an essentially exact solution of the problem which is in a form suitable for evaluation on small desk-top computers. It is shown that for a circular base orbit at 1 a.u., a 15 kw solar electric system with a specific impulse of 3000 sec is capable of delivering an 1100 kg net payload to a heliographic latitude of about 63°.

Nomenclature

a	= semi major axis
c	= dimensionless thruster exhaust velocity, Eq. (10)
e	= eccentricity
f	= duty cycle, fraction of orbit with thrust on
h	= function defined by Eq. (49)
i	= inclination
m_0	= spacecraft mass (kg)
m_p	= dimensionless propellant mass expended
n	= integer
t	= dimensionless time t/t_1
T_1	= time scale proportional to unperturbed orbit period
T_2	= time scale based on low-thrust acceleration
ν	= true anomaly
Ω	= longitude of ascending node
ω	= argument of perihelion
ϵ	= ratio of initial thrust acceleration to gravitational acceleration
ξ	= steering function

Introduction

A potential application of solar electric low-thrust propulsion is to place a scientific payload at a high heliographic latitude for studies of solar phenomena. The purpose of the analysis presented here is to develop a simple analytic model that can be used to rapidly and accurately predict spacecraft motion and performance for design of this mission. The two-variable expansion method is employed to generate the solution of the nonlinear variational trajectory equations in terms of asymptotic series. Only steering programs of practical interest are evaluated. Although no attempt is made to optimize the trajectories, it can be demonstrated that the departure from optimum performance is not large.

It is shown that for circular base orbits at 1 a.u., a 15 kw solar electric system with a 3000 sec specific impulse is capable of delivering an 1100 kg net payload to a heliographic latitude of about 63°. Since the solutions are developed in analytic form, it is possible to make all trajectory calculations on small desk-top computers. Extensive parameters studies and performance calculations are facilitated in this way.

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*Member of Technical Staff, Mission Analysis Division, Jet Propulsion Laboratory, and Professor of Mechanical Engineering, University of Utah. Member AIAA.

Analysis

Since the motion of the spacecraft is characterized by a slow evolution of the orbital elements due to the low magnitude of the thrust as compared to the solar gravitation, it is appropriate to describe the motion with an osculating conic. Thrust is assumed to be applied only normal to the instantaneous orbit plane; orbit size and eccentricity do not change. The pertinent variational equations are, therefore

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \left[\frac{\cos(\nu+\omega)}{ua^{1/2}(1-e^2)^{1/2}}\right]a_n \tag{1}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \left[\frac{\sin(\nu+\omega)}{ua^{1/2}(1-e^2)^{1/2}\sin i}\right]a_n \tag{2}$$

$$\mathrm{d}\omega/\mathrm{d}t = [-a^{1/2}(1-e^2)^{1/2}\sin(\nu+\omega)\cot i]a_n \tag{3}$$

$$\mathrm{d}\nu/\mathrm{d}t = a^{1/2}(1-e^2)^{1/2}u^2 - \mathrm{d}\omega/\mathrm{d}t - \cos i\,\mathrm{d}\Omega/\mathrm{d}t \tag{4}$$

where ν is the true anomaly, a is the semi-major axis, e is the eccentricity of the osculating ellipse, i is the inclination of the orbit with respect to the solar equator, Ω is the longitude of the ascending node, ω is the argument of the perihelion, and u is the inverse of the radial distance from the sun and is given by

$$u(t) = (1+e\cos\nu)/a(1-e^2) \tag{5}$$

The dimensionless variables employed in Eqs. (1-5) are

$$a = a'/r_0, \quad u = r_0 u', \quad t = t'/T_1,$$

$$\text{and } m' = m/m_0 \tag{6}$$

where primes denote dimensional quantities; subscript 0 refers to conditions at the initial point of the orbit. The time scale $T_1 = (r_0^3/GM)^{1/2}$ is proportional to the period of the unperturbed orbit, and r_0 is the radial distance at which thrust commences. Time is also measured from this point. The normal thrust acceleration may be expressed as

$$a_n = [\epsilon u^2/(1-m_p)]\xi \quad (7)$$

where

$$\epsilon = \frac{F_0'/m_0'}{GM/(r_0')^2} \quad (8)$$

is the ratio of the initial thrust acceleration to initial gravitational acceleration. ϵ is of the order of 10^{-2} for present solar electric systems and thus is suitable for use as a perturbation parameter in constructing asymptotic solutions. The parameter $\xi(t)$ represents the throttling of the thrust output; the sign of ξ determines the sense of the thrust output; $|\xi|=1$ corresponds to a nominal inverse square dependence of thruster output on solar distance; m_p is the dimensionless propellant mass consumed in reaching a given point on the orbit. The mass flow rate can be written in terms of ξ as

$$\mathrm{d}m_p/\mathrm{d}t = (\epsilon/c)u^2|\xi| \quad (9)$$

where

$$c = gI_{SP}/(GM/r_0)^{1/2} \quad (10)$$

is the dimensionless thruster exhaust velocity; I_{SP} is the specific impulse, and g is nominal gravitational acceleration at the surface of the earth used in the definition of I_{SP}. c is typically about 1 for low-thrust systems of current interest.

Solution by the Two-Variable Method

Since there are two characteristic time scales

$$T_1 = (r_0^3/GM)^{1/2} \quad (11)$$

and

$$T_2 = [r_0/F_0'/m_0')]^{1/2} \quad (12)$$

inherent in the problem, simple perturbation solutions cannot be used. Such solutions would only be valid for times of the order of ϵ^{-1}; much longer flight times are required in low-thrust missions. This mathematical difficulty can be circumvented by use of the two-variable expansion method.[1] Other methods are available. For example the method of averaging produces equivalent solutions.[2] It is convenient to employ the true anomaly ν as the independent variable. Equations (1-3) can be then rewritten by using Eq. (4) and noting that $\mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}\Omega/\mathrm{d}t$ in Eq. (4) are of the order ϵ. Thus

$$\frac{\mathrm{d}t}{\mathrm{d}\nu} = \frac{[a(1-e^2)]^{-1/2}}{u^2} + 0(\epsilon) \quad (13)$$

$$\frac{\mathrm{d}i}{\mathrm{d}\nu} = \frac{\epsilon}{(1-m_p)}\frac{\cos(\nu+\omega)}{(1+e\cos\nu)}\xi + 0(\epsilon^2) \quad (14)$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\nu} = \frac{\epsilon}{(1-m_p)}\frac{\sin(\nu+\omega)}{(1+e\cos\nu)\sin i}\xi + 0(\epsilon^2) \quad (15)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}\nu} = \frac{\epsilon}{(1-m_p)}\frac{\sin(\nu+\omega)}{\tan i}\xi + 0(\epsilon^2) \quad (16)$$

Application of the two-variable method requires the definition of two suitable independent variables. Following (1), define

$$\nu^* = \nu = \text{FAST VARIABLE} \quad (17)$$

$$\tilde{\nu} = \epsilon\nu = \text{SLOW VARIABLE} \quad (18)$$

and expand each osculating element as a perturbation series in ϵ. For example,

$$i(\epsilon,\nu^*,\tilde{\nu}) = i^{(0)}(\epsilon,\nu^*,\tilde{\nu}) + \epsilon i^{(1)}(\epsilon,\nu^*,\tilde{\nu}) + 0(\epsilon^2) \quad (19)$$

Inserting the various expansions into Eqs. (14-16) and collecting like powers of ϵ yields the following sets of equations

$$0(1): \frac{\partial i(0)}{\partial \nu^*} = \frac{\partial \Omega^{(0)}}{\partial \nu^*} = \frac{\partial \omega^{(0)}}{\partial \nu^*} = 0 \quad (20)$$

$$0(\epsilon): \frac{\partial i^{(1)}}{\partial \nu^*} = -\frac{\partial i^{(0)}}{\partial \tilde{\nu}} + \frac{\cos(\nu+\omega)\xi}{(1-m_p)(1+e\cos\nu)} \quad (21)$$

$$\frac{\partial \Omega^{(1)}}{\partial \nu^*} = -\frac{\partial \Omega^{(0)}}{\partial \tilde{\nu}} + \frac{\sin(\varphi+\omega)\xi}{(1-m_p)(1+e\cos\nu)\sin i^{(0)}} \quad (22)$$

$$\frac{\partial \omega^{(0)}}{\partial \nu^*} = -\frac{\partial \omega^{(0)}}{\partial \tilde{\nu}} - \frac{\sin(\nu+\omega)\xi}{(1-m_p)\tan i^{(0)}} \quad (23)$$

and so on for higher orders of ϵ. The trigonometric terms, for example $\sin i$, are handled as follows:

$$\sin i = \sin(i^{(0)} + \epsilon i^{(1)} + \ldots)$$

$$= \sin i^{(0)} \cos(\epsilon i^{(1)} + \ldots) + \cos i^{(0)} \times \sin(\epsilon i^{(1)} + \ldots)$$

$$\cong \sin i^{(0)} + \epsilon i^{(1)} \cos i^{(0)} + 0(\epsilon^2) \quad (24)$$

Thus only the leading terms appear in the order (ϵ) Eqs. (22) and (23). The propellant expended must be determined by integrating Eq. (9). Thus, using Eqs. (9) and (13)

$$\mathrm{d}m_p/\mathrm{d}\nu = \epsilon|\xi|/[ca^{1/2}(1-e^2)^{1/2}] \quad (25)$$

It is clear from Eq. (20) that the leading terms in the series for the osculating elements are functions only of the flow variable $\tilde{\nu}$. This corresponds to the physical fact that the motion is a slow modification of the osculating variables resulting from the low-thrust acceleration. To determine $i^{(0)}$, $\omega^{(0)}$, and $\Omega^{(0)}$ requires use of the order ϵ Eqs. (21-23) and (25). This is accomplished by requiring that there be no secular variations which would invalidate the assumed asymptotic behavior of the variables. All terms on the right of the $0(\epsilon)$ equations not producing purely oscillatory variations in ν^* must be suppressed. This process yields a set of equations of the form

$$\partial i^{(0)}/\partial\tilde{\nu} = f_1, \quad \partial\Omega^{(0)}/\partial\tilde{\nu} = f_2, \quad \partial\omega^{(0)}/\partial\tilde{\nu} = f_3 \quad (26)$$

where the f_n depend on the steering function ξ. ξ, m_p, and the function $(1+e\cos\nu)$, which appears in Eqs. (21) and (22), must be expressed as Fourier series to facilitate suppression of the secular terms. The series for ξ and m_p depend on the specific thrust program under investigation; $(1+e\cos\nu)^{-1}$ can be expanded as

$$(1+e\cos\nu)^{-1} = (1-e^2)^{-1/2}\left\{1+2\sum_{n=1}^{\infty}\left[\frac{(1-e^2)^{1/2}-1}{e}\right]^n \cos n\nu^*\right\} \quad (27)$$

Thrust Program

Further progress requires specification of the steering function $\epsilon(\nu^*, \tilde{\nu})$. Only practical thrust programing is investigated here; the departure from optimal thrusting is not large. Since

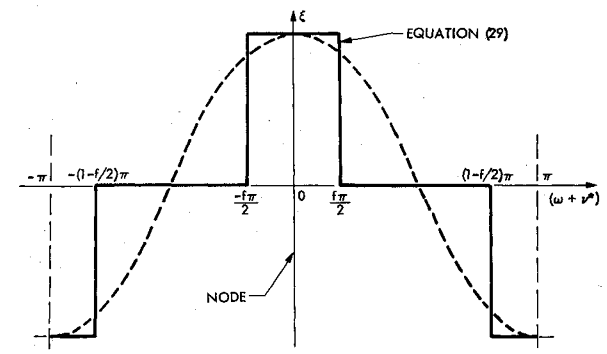

Fig. 1 Assumed out of ecliptic thrusting program.

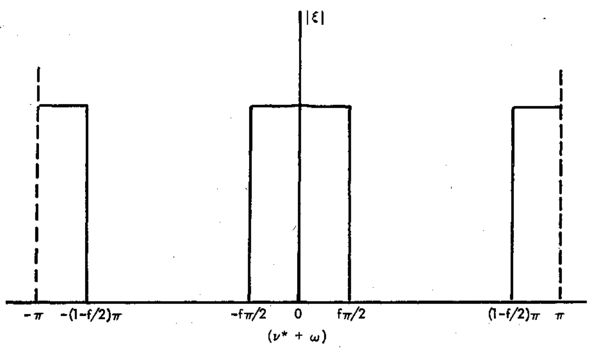

Fig. 2 |ξ| vs position relative to node.

it is desired to make the final heliographic inclination as large as possible, ξ should be chosen to make $\partial i(0)/\partial \nu$ as large as possible. Referring to Eq. (21), it is clear that a function of the form

$$\xi \approx \cos(\nu+\omega) \tag{28}$$

tends to maximize the nonoscillatory part of the last terms in Eq. (21). Figure 1 shows the ξ of Eq. (28) and a modification of this thrust program, which is more easily mechanized in an actual spacecraft system. Note that the thrusting is symmetrical about the nodes of the trajectory, and that thrust direction must change 180° during each orbit. The modified thrust program requires no throttling and is therefore more readily utilized in practice. Parameter f in Fig. 1 represents the angular fraction of each orbit during which the thruster operates. A Fourier series for this steering program is

$$\xi(\nu^*) = \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\sin f(2m-1)\pi/2}{(2m-1)} \cos(2m-1)(\nu^*+\omega) \tag{29}$$

f will be referred to in what follows as the "duty cycle" of the low-thrust system, although here it is expressed in terms of angular fraction instead of time as in the usual application of this terminology. This steering program is evaluated in the following sections for different base orbit geometries.

Circular Base Orbit

An orbital geometry of practical interest is one that remains in a nearly circular path at one astronomical unit as the inclination is raised by the low-thrust system. Such a trajectory minimizes communication distance and simplifies scientific interpretation of data by eliminating the solar distance as a variable. Since some detailed studies of this case have been performed using integrating trajectory programs, it will be possible to assess the validity of the present model by direct comparison.

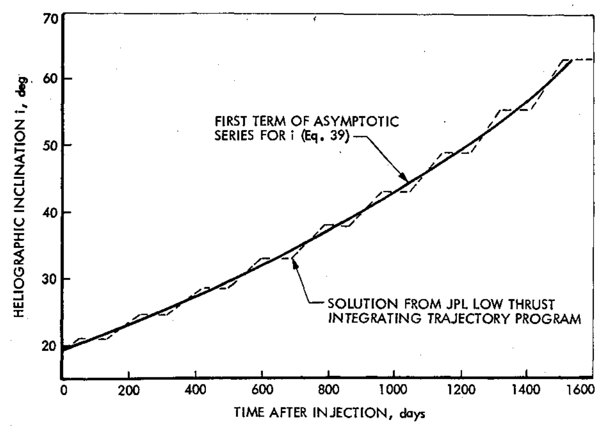

Fig. 3 Comparison of one-term asymptotic solution to integrated solution.

For the circular case, $a=1$, the governing equations simplify to

$$\frac{\partial i^{(0)}}{\partial \bar{\nu}} = \left\langle \frac{\xi\cos(\nu^*+\mathrm{m})}{(1-m_p)} \right\rangle \tag{30}$$

$$\frac{\partial \Omega^{(0)}}{\partial \bar{\nu}} = \left\langle \frac{\xi\sin(\nu^*+\omega)}{(1-m_p)\sin i^{(0)}} \right\rangle \tag{31}$$

$$\frac{\partial \omega^{(0)}}{\partial \bar{\nu}} = \left\langle \frac{-\xi\sin(\nu^*+\omega)}{(1-m_p)\tan i^{(0)}} \right\rangle \tag{32}$$

where the triangular brackets refer to the nonharmonic part of the enclosed function. Using ξ given by the Eq. (29), the spacecraft mass $(1-m_p)$ is found by integrating Eq. (25)

$$dm_p/d\nu^* = \epsilon|\xi|/c \tag{33}$$

where $|\xi|$ is best expressed as a Fourier series. Figure 2 shows $|\xi|$ vs $(\nu^*+\omega)$. The Fourier series is

$$|\xi| = f + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin m f\pi}{m}\cos 2m(\nu^*+\omega) \tag{34}$$

and the dimensionless spacecraft mass is

$$(1-m_p) = 1 - \frac{f\bar{\nu}}{c} - \frac{\epsilon}{\pi c}\sum_{m=1}^{\infty}\frac{\sin m f\pi}{m^2}\sin 2m(\nu^*+\mathrm{m}) \tag{35}$$

Inserting Eqs. (29) and (35) into Eqs. (30-32) yields

$$\frac{\partial i^{(0)}}{\partial \bar{\nu}} = \frac{2\sin f\pi/2}{\pi(1-f\bar{\nu}/c)} \tag{36}$$

$$\partial \Omega^{(0)}/\partial \bar{\nu} = 0 \tag{37}$$

$$\partial \omega^{(0)}/\partial \bar{\nu} = 0 \tag{38}$$

which indicates that to first order, the normal thrusting symmetrical about the nodes generates no change in the positions of the node or perihelion. Equation (36) is easily integrated to give

$$i^{(0)} = i_0 - (2c/f\pi)\sin(f\pi/2)\ln(1-(\epsilon f/c)\nu) + O(\epsilon) \tag{39}$$

where i_0 is the heliographic inclination at start of low-thrust burn. Since the base orbit is circular, dimensionless time can be expressed as $t=\nu$ in evaluating Eq. (39). Figure 3 is a plot of Eq. (39) corresponding to an initial spacecraft mass of 2648 kg, $I_{SP}=3000$ sec, flight time of 1510 days and final spacecraft mass of 1100 kg. The solution is superimposed on results from the JPL low-thrust integrating program for comparison. It will be noted that Eq. (39) gives a very good representation of the performance of the system, but does not describe the details of the path; it will be shown presently that retention of another term in the series corrects the path representation. Nevertheless, if all that is required is information concerning performance of the system in terms of final heliographic inclination for a given set of parameters, Eq. (39) gives a very useful approximation. The initial inclination of $i_0 = 19.3°$ in Fig. 3 corresponds to the ballistic performance of a Tital IIIE/Centaur with a 2648 kg injected mass and includes the 7.2° inclination of the solar equator to the ecliptic plane.

Correction to 0 (ε)

The mathematical process demonstrated above can be continued to determine more terms in the asymptotic series to yield a more accurate representation of the motion. To determine the oscillatory part of $i^{(1)}$, Eq. (21) is utilized; the parts which would have given rise to incorrect secular terms were suppressed in determining $i^{(0)}$. Thus $i^{(1)}$ is found by integrating

$$\partial i^{(1)}/\partial \nu^* = \xi \cos(\nu+\omega)/(1-m_p) \tag{40}$$

where only the harmonic part of the right-hand side is involved. That is

$$\frac{\partial i^{(1)}}{\partial \nu^*} = \frac{4}{\pi(1-m_p)} \left\{ \frac{\sin f\pi/2}{2} \cos 2(\nu^*+\omega) - \sum_{m=1}^{\infty} \frac{\sin f(2m-1)\pi/2}{(2m-1)} \cos(\nu^*+\omega)\cos(2m-1)(\nu^*+\omega) \right\} \tag{41}$$

Integrating, using the facts that

$$\partial i^{(1)}/\partial \nu^* = (1+\partial\omega/\partial\nu^*)\ \partial i^{(1)}/[\partial(\nu^*+\omega)] \tag{42}$$

and

$$\partial\omega/\partial\nu^* \text{ is of } 0\ (\epsilon)$$

one finds

$$i^{(1)} = 1/[\pi(1-m_p)] \left\{ \sin f\pi/2 \sin 2(\nu+\omega) + \sum_{m=2}^{\infty} \frac{\sin f(2m-1)\pi/2}{(2m-1)} \times \left[\frac{\sin 2(m-1)(\nu+\omega)}{m-1} + \frac{\sin(2m)(\nu+\omega)}{m} \right] \right\} \tag{43}$$

$i^{(1)}$ does not have a nonoscillatory part; it is easily demonstrated that

$$\partial i^{(1)}/\partial\bar{\nu} = 0 \text{ and } \partial i^{(2)}/\partial\nu^* = 0 \tag{44}$$

Thus the solution

$$i = i^{(0)} + \epsilon i^{(1)} \tag{45}$$

where $i^{(0)}$ and $i^{(1)}$ are from Eqs. (39) and (43) respectively, should represent a nearly exact model of the spacecraft motion. Figure 4 is a plot of Eq. (45) for the same conditions as in the previous example ($m_0 = 2648$ kg, $m_f = 1100$ kg, $I_{SP} = 3000$ sec, and $f = 0.5592$.) Under these conditions, $\epsilon = 0.39$, $\dot{m}_0 = 0.208 \cdot 10^{-4}$ kg/sec, and $c = 0.987$. Comparison of Fig. 4 to the numerical solution shown in Fig. 3 (dashed curve) shows essentially exact agreement. The results shown in the figure were computed carrying only 5 terms of the series; all infinite series used in the solutions converge rapidly. Thus the solution given in Eq. (45) is a very good model of the orbital path as well as an accurate estimate of performance. An advantage of the present method is that Eq. (45) can be easily programed for use on a desktop calculator. The results shown in Fig. 3 and 4 were generated using this procedure.

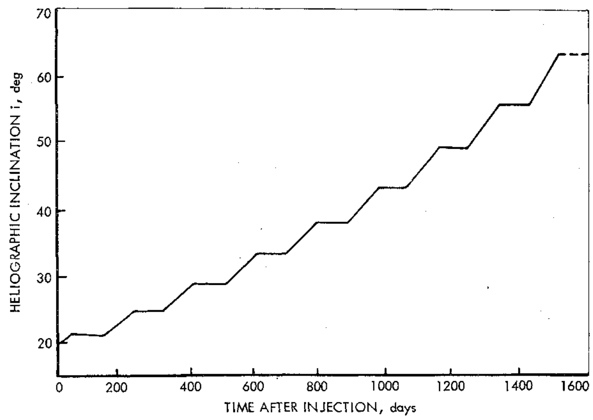

Fig. 4 Plot of two-term asymptotic series for heliographic inclination.

Elliptical Base Orbit

It has been suggested that if part of the injection energy from the launch vehicle is used to make the base orbit elliptical, the low-thrust system is capable of producing a larger change of inclination. This is an approximation to the more general case wherein the low-thrust propulsion is used to change the orbit size and shape and inclination simultaneously.

For the elliptical case, integration of Eq. (25) yields the following expression for the spacecraft mass

$$(1-m_p) = 1 - \frac{[a(1-e^2)]^{-1/2}}{c} \left[f\bar{\nu} + \frac{\epsilon}{\pi} \sum_{n=1}^{\infty} \frac{\sin n f\pi}{n^2} \sin 2n(\nu^*+\omega) \right] \tag{46}$$

and

$$\frac{\partial i^{(0)}}{\partial\bar{\nu}} = \frac{2}{\pi(1-e^2)^{1/2}} \left[1 - \frac{f a^{-1/2}\bar{\nu}}{c(1-e^2)^{1/2}} \right]^{-1} \times \left\{ \sin\frac{f\pi}{2} + \tfrac{1}{2} \sum_{n-1}^{\infty} \left[\frac{\sin(2n-1)f\pi/2}{(2n-1)} + \frac{\sin(2n+1)f\pi/2}{(2n+1)} \right] \right\} \times [((1-e^2)^{1/2}-1)/3]^{2n} \cos 2n\omega_0 \tag{47}$$

Integrating Eq. (47),

$$i^{(0)} = i_0 - c a^{1/2} \frac{h(f)}{f} \ln\left\{ 1 - \left[\frac{f}{c a^{1/2}(1-e^2)^{1/2}} \right] \nu \right\} \tag{48}$$

where

$$h(f) = \frac{2}{\pi} \left\{ \sin f\pi/2 + \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(2n-1)f\pi/2}{(2n-1)} + \left[\frac{\sin(2n+1)f\pi/2}{(2n+1)} \right] \left[\frac{(1-e^2)^{1/2}-1}{e} \right]^{2n} \times \cos 2n\omega_0 \right\} \tag{49}$$

ω_0 is the initial argument of the perihelion.

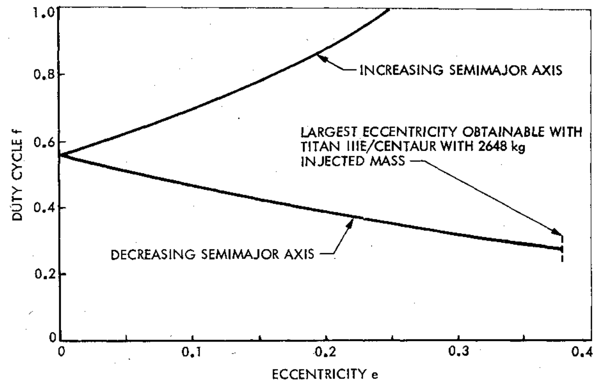

Fig. 5 Duty cycle vs eccentricity for flight time of 1510 days and 1100 kg final mass.

Time may be calculated by using Eq. (13). Thus

$$dt = \{[a(1-e^2)]^{3/2}/(1+e\cos\nu)^2\}d\nu + O(\epsilon) \tag{50}$$

Inserting a Fourier series for $(1+e\cos\nu)^{-2}$,

$$t = \int_0 [a(1-e^2)]^{3/2}\left[\frac{b_{20}}{2} + \sum_{n=1}^{\infty} b_{2n}\cos n\nu\right]d\nu \tag{51}$$

where

$$b_{20} = \frac{2}{(1-e^2)^{3/2}}, \quad b_{2n} = \frac{2}{(1-e^2)^{3/2}}$$

$$\times\left[\frac{(1-e^2)^{1/2}-1}{e}\right]^n \times \left[1+n(1-e^2)^{1/2}\right] \tag{52}$$

Carrying out the integration yields

$$t = a^{3/2}\left\{\nu + 2\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{(1-e^2)^{1/2}-1}{e}\right]^n\right.$$

$$\left.\times\left[1+n(1-e^2)^{1/2}\right]\sin\nu\right\} \tag{53}$$

Since the Fourier series converges rapidly, only a few terms need be carried to achieve good accuracy. This is true of all infinite series employed in this paper.

The example conditions used in the previous section to illustrate the solution for a circular base orbit are employed to assess the effects of orbit size and shape on the inclination change. To provide a useful comparison to the circular case, the same flight time of 1510 days is assumed. This requires that the correct value of f be used for each value of initial eccentricity. Figure 5 shows f vs e for the range of values attainable with the Tital IIIE/Centaur launch vehicle and 2648 kg injected mass. f is found by first determining the true anomaly corresponding to the desired final time, using Eq. (53). This result, along with the desired final mass of 1100 kg, is used in Eq. (46) to determine the value of f for each value of eccentricity. The initial eccentricity attainable at injection is found from

$$i_0 = \sin^{-1}\{r_0 v_{HL}^2/GM - [(1\pm e)^{1/2} - 1]^2\}^{1/2} \tag{54}$$

where the upper sign is for increasing semi-major axis and the lower sign for decreasing a; r_0 is the heliocentric radius at injection, and v_{HL} is the injection velocity. It is assumed in the following calculations that the high-thrust maneuver takes place at the node of the orbit. For the assumed spacecraft initial mass, the maximum orbit size is $a=1.872$ a.u. with $e=0.466$, the minimum size is $a=0.726$ a.u. with $e=0.377$. Figure 6 shows the initial high-thrust inclination change corresponding to each eccentricity. The limiting value of 7.2° is the inclination of the ecliptic plane in the heliocentric equatorial coordinate system.

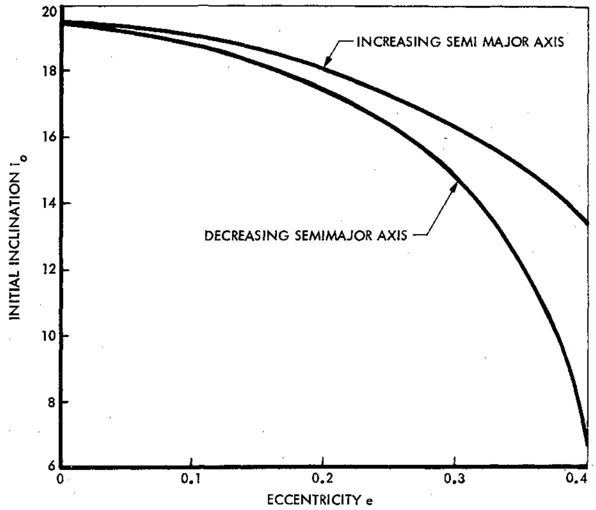

Fig. 6 Initial heliographic inclination vs eccentricity of base orbit.

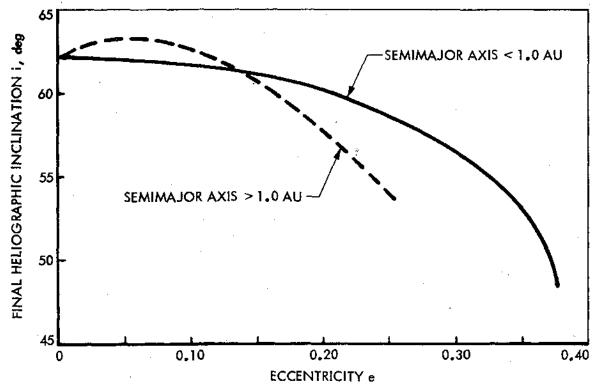

Fig. 7 Final inclination vs *e*.

Evaluating Eq. (48) for the assumed conditions yields the results plotted in Fig. 7. Decreasing the orbit size decreases the attainable inclinations; increasing the size of the orbit initially increases the final inclination. For the parameters assumed for this example, a gain of about 2° is possible at an eccentricity of $e=0.05$ and $a=1.05$ a.u. Further gains are probably possible if the low-thrust system itself is used to make the base orbit modifications.

It has been suggested that the inclination could be further increased by gravity assist using a flyby of Venus; calculations show that an inclination change of about $\Delta i = 5°$ could be achieved. However, as Fig. 7 shows, about 2° of this gain would be lost; the elliptical transfer trajectory would require an eccentricity of about 0.16 and a decrease in orbit size is obviously involved.

Conclusions

It has been demonstrated that the first two terms of an asymptotic series are capable of accurately representing the motion of a low-thrust space vehicle with thrusting normal to the plane of the osculating orbit. Although optimization of the thrust program was not attempted here, it appears that an optimization scheme could be incorporated by representing the steering function as an unspecified Fourier series; the coefficients could then be adjusted using variational techniques to maximize the inclination angle. The two-variable method was found to be readily applicable in the present problem. Other similar method would no doubt yield equivalent solutions; only slight differences in procedure and mathematical complexity would be introduced by using the method of averaging, for example, instead. The results compare closely to solutions generated by integrating low-thrust trajectory programs. The present solutions can be used on programable desktop calculators to produce required mission analysis data and parameter tradeoffs efficiently and rapidly.

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